

# On Generalized Sierpiński Graphs

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## Abstract

In this paper we obtain closed formulae for several parameters of generalized Sierpiński graphs  $S(G, t)$  in terms of parameters of the base graph  $G$ . In particular, we focus on the chromatic, vertex cover, clique and domination numbers.

*Keywords:* Sierpiński graphs; vertex cover number; independence number; chromatic number; domination number.

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## 1 Introduction

Let  $G = (V, E)$  be a non-empty graph of order  $n$ . We denote by  $V^t$  the set of words of size  $t$  on alphabet  $V$ . The letters of a word  $u$  of length  $t$  are denoted by  $u_1u_2\dots u_t$ . The concatenation of two words  $u$  and  $v$  is denoted by  $uv$ . Klavžar and Milutinović introduced in [8] the graph  $S(K_n, t)$  whose vertex set is  $V^t$ , where  $\{u, v\}$  is an edge if and only if there exists  $i \in \{1, \dots, t\}$  such that:

- (i)  $u_j = v_j$ , if  $j < i$ ; (ii)  $u_i \neq v_i$ ; (iii)  $u_j = v_i$  and  $v_j = u_i$  if  $j > i$ .

When  $n = 3$ , those graphs are exactly Tower of Hanoi graphs. Later, those graphs have been called Sierpiński graphs in [9] and they were studied by now from numerous

points of view. The reader is invited to read, for instance, the following recent papers [5, 6, 7, 9, 10, 11] and references therein. This construction was generalized in [4] for any graph  $G = (V, E)$ , by defining the  $t$ -th *generalized Sierpiński graph* of  $G$ , denoted by  $S(G, t)$ , as the graph with vertex set  $V^t$  and edge set defined as follows.  $\{u, v\}$  is an edge if and only if there exists  $i \in \{1, \dots, t\}$  such that:

- (i)  $u_j = v_j$ , if  $j < i$ ; (ii)  $u_i \neq v_i$  and  $\{u_i, v_i\} \in E$ ; (iii)  $u_j = v_i$  and  $v_j = u_i$  if  $j > i$ .

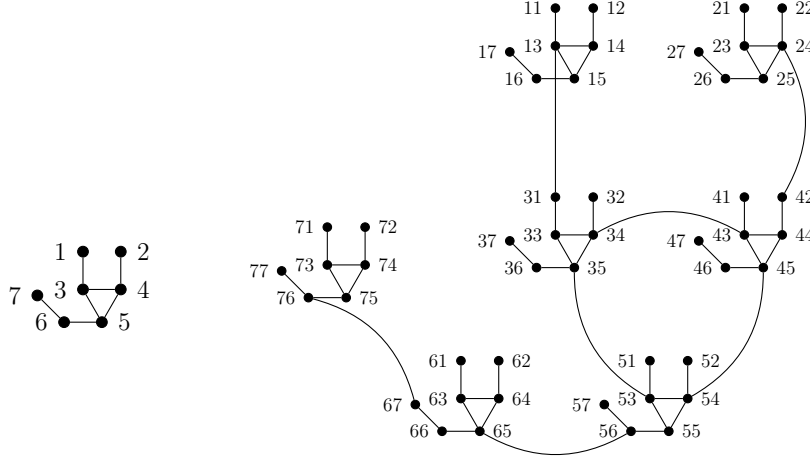


Figure 1: A graph  $G$  and the generalized Sierpiński graph  $S(G, 2)$

Figure 1 shows a graph  $G$  and the generalized Sierpiński graph  $S(G, 2)$ , while Figure 2 shows the Sierpiński graph  $S(G, 3)$ .

Notice that if  $\{u, v\}$  is an edge of  $S(G, t)$ , there is an edge  $\{x, y\}$  of  $G$  and a word  $w$  such that  $u = wxyy\dots y$  and  $v = wyxx\dots x$ . In general,  $S(G, t)$  can be constructed recursively from  $G$  with the following process:  $S(G, 1) = G$  and, for  $t \geq 2$ , we copy  $n$  times  $S(G, t - 1)$  and add the letter  $x$  at the beginning of each label of the vertices belonging to the copy of  $S(G, t - 1)$  corresponding to  $x$ . Then for every edge  $\{x, y\}$  of  $G$ , add an edge between vertex  $xyy\dots y$  and vertex  $yx\dots x$ . See, for instance, Figure 2. Vertices of the form  $xx\dots x$  are called *extreme vertices* of  $S(G, t)$ . Notice that for any graph  $G$  of order  $n$  and any integer  $t \geq 2$ ,  $S(G, t)$  has  $n$  extreme vertices and, if  $x$  has degree  $d(x)$  in  $G$ , then the extreme vertex  $xx\dots x$  of  $S(G, t)$  also has degree  $d(x)$ . Moreover, the degrees of two vertices  $yx\dots x$  and  $xyy\dots y$ , which connect two copies of  $S(G, t - 1)$ , are equal to  $d(x) + 1$  and  $d(y) + 1$ , respectively.

For any  $w \in V^{t-1}$  and  $t \geq 2$  the subgraph  $\langle V_w \rangle$  of  $S(G, t)$ , induced by  $V_w = \{wx : x \in V\}$ , is isomorphic to  $G$ . Notice that there exists only one vertex  $u \in V_w$  of the form  $w'xx\dots x$ , where  $w' \in V^r$  for some  $r \leq t - 2$ . We will say that  $w'xx\dots x$  is the *extreme vertex* of  $\langle V_w \rangle$ , which is an extreme vertex in  $S(G, t)$  whenever  $r = 0$ . By definition of  $S(G, t)$  we deduce the following remark.

**Remark 1.** Let  $G = (V, E)$  be a graph, let  $t \geq 2$  be an integer and  $w \in V^{t-1}$ . If  $u \in V_w$  and  $v \in V^t - V_w$  are adjacent in  $S(G, t)$ , then either  $u$  is the extreme vertex of  $\langle V_w \rangle$  or  $u$  is adjacent to the extreme vertex of  $\langle V_w \rangle$ .

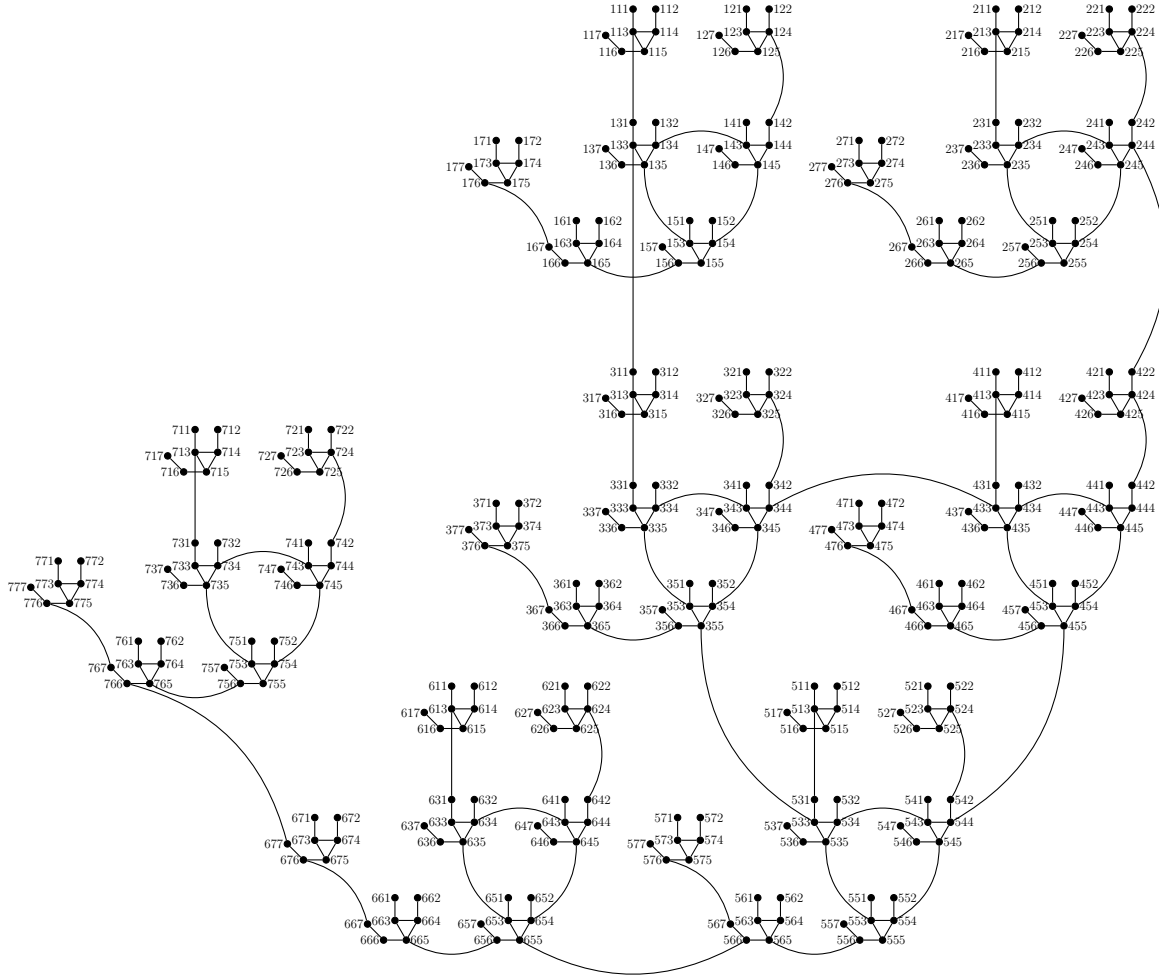


Figure 2: The generalized Sierpiński graph  $S(G, 3)$ . The base graph  $G$  is shown in Figure 1.

To the best of our knowledge, [13] is the first published paper studying the generalized Sierpiński graphs. In that article, the authors obtained closed formulae for the Randić index of polymeric networks modelled by generalized Sierpiński graphs. Also, the total chromatic number of generalized Sierpiński graphs has recently been studied in [3]. In this paper we obtain closed formulae for several parameters of generalized Sierpiński graphs  $S(G, t)$  in terms of parameters of the base graph  $G$ . In particular, we focus on the chromatic, vertex cover, clique and domination numbers.

## 2 Some Remarks on Trees

Given a graph  $G$ , the order and size of  $S(G, t)$  is obtained in the following remark.

**Remark 2.** *Let graph  $G$  be a graph of order  $n$  and size  $m$ , and let  $t$  be a positive integer. Then the order of  $S(G, t)$  is  $n^t$  and the size is  $m \frac{n^t - 1}{n - 1}$ .*

*Proof.* By definition of  $S(G, t)$ , for any  $t \geq 2$  we have that the order of  $S(G, t)$  is  $\mathcal{O}(S(G, t)) = n\mathcal{O}(S(G, t - 1))$  and  $\mathcal{O}(S(G, 1)) = n$ . Hence,  $\mathcal{O}(S(G, t)) = n^t$ . Analogously, the size of  $S(G, t)$  is  $\mathcal{S}(S(G, t)) = n\mathcal{S}(S(G, t - 1)) + m$  and  $\mathcal{S}(S(G, 1)) = m$ . Then  $\mathcal{S}(S(G, t)) = m(n^{t-1} + n^{t-2} + \dots + 1) = m \frac{n^t - 1}{n - 1}$ .  $\square$

**Corollary 3.** *For any tree  $T$  and any positive integer  $t$ ,  $S(T, t)$  is a tree.*

*Proof.* Let  $n$  be the order of  $T$ . By the connectivity of  $T$  we have that  $S(T, t)$  is connected. On the other hand, by Remark 2,  $S(T, t)$  has order  $n^t$  and size  $n^t - 1$ . Therefore, the result follows.  $\square$

The next result gives a formula for the number of leaves in a generalized Sierpiński tree. A vertex with degree one in a tree  $T$  is called a *leaf* and a vertex adjacent to a leaf is called a *support*. The number of leaves of a tree  $T$  will be denoted by  $\varepsilon(T)$  and the set of support vertices of  $T$  by  $\text{Sup}(T)$ . Also, if  $x \in \text{Sup}(T)$ , then  $\varepsilon_T(x)$  will denote the number of leaves of  $T$  which are adjacent to  $x$ .

**Theorem 4.** *Let  $T$  be a tree of order  $n$  having  $\varepsilon(T)$  leaves. For any positive integer  $t$ , the number of leaves of  $S(T, t)$  is*

$$\varepsilon(S(T, t)) = \frac{\varepsilon(T)(n^t - 2n^{t-1} + 1)}{n - 1}.$$

*Proof.* Let  $t \geq 2$ . For any  $x \in V$ , we denote by  $S_x(T, t - 1)$  the copy of  $S(T, t - 1)$  corresponding to  $x$  in  $S(T, t)$ , i.e.,  $S_x(T, t - 1)$  is the subgraph of  $S(T, t)$  induced by the set  $\{xw : w \in V^{t-1}\}$ , which is isomorphic to  $S(T, t - 1)$ . To obtain the result, we only need to determine the contribution of  $S_x(T, t - 1)$  to the number of leaves of  $S(T, t)$ , for all  $x \in V$ . By definition of  $S(T, t)$ , there exists an edge of  $S(T, t)$  connecting the vertex  $xy\dots y$  of  $S_x(T, t - 1)$  with the vertex  $yx\dots x$  of  $S_y(T, t - 1)$  if and only if  $x$  and  $y$  are adjacent in  $T$ . Hence, a leaf  $xy\dots y$  of  $S_x(S(T, t - 1))$  is adjacent in  $S(T, t)$  to a vertex  $yx\dots x$  of  $S_y(T, t - 1)$  if and only if  $y$  is a leaf of  $T$  and  $x$  is its support vertex. Thus, if  $x \in \text{Sup}(T)$ , then the contribution of  $S_x(T, t - 1)$  to the number of leaves of  $S(T, t)$  is  $\varepsilon(S(T, t - 1)) - \varepsilon_T(x)$  and, if  $x \notin \text{Sup}(T)$ , then the contribution of  $S_x(T, t - 1)$  to the number of leaves of  $S(T, t)$  is  $\varepsilon(S(T, t - 1))$ . Then we obtain,

$$\begin{aligned} \varepsilon(S(T, t)) &= (n - |\text{Sup}(T)|)\varepsilon(S(T, t - 1)) + \sum_{x \in \text{Sup}(T)} (\varepsilon(S(T, t - 1)) - \varepsilon_T(x)) \\ &= n\varepsilon(S(T, t - 1)) - \varepsilon(T). \end{aligned}$$

Now, since  $\varepsilon(S(T, 1)) = \varepsilon(T)$ , we have that

$$\varepsilon(S(T, t)) = \varepsilon(T) (n^{t-1} - n^{t-2} - \dots - n - 1) = \varepsilon(T) \left( n^{t-1} - \frac{(n^{t-1} - 1)}{n - 1} \right).$$

Therefore, the result follows.  $\square$

### 3 Chromatic Number and Clique Number

The *chromatic number* of a graph  $G = (V, E)$ , denoted by  $\chi(G)$ , is the smallest number of colors needed to color the vertices of  $G$  so that no two adjacent vertices share the same color. A *proper vertex-colouring* of  $G$  is a map  $f : V \rightarrow \{1, 2, \dots, k\}$  such that for any edge  $\{u, v\}$  of  $G$ ,  $f(u) \neq f(v)$ . The elements of  $\{1, 2, \dots, k\}$  are called colours, the vertices of one colour form a *colour class* and we say that  $f$  is a *k-colouring*. So the chromatic number of  $G$  is the minimum  $k$  such that there exists a  $k$ -colouring. For instance, for any bipartite graph  $G$ ,  $\chi(G) = 2$ . Since every tree is a bipartite graph, by Corollary 3 we conclude that for any tree  $T$  and any positive integer  $t$ ,  $\chi(S(T, t)) = 2$ .

The problem of finding chromatic number of a graph is NP-hard, [2]. This suggests finding the chromatic number for special classes of graphs or obtaining good bounds on this invariant. As shown in [12],  $\chi(K_n, t) = n$ . We shall show that the chromatic number of a generalized Sierpiński graph is determined by the chromatic number of its base graph.

**Theorem 5.** *For any graph  $G$  and any positive integer  $t$ ,*

$$\chi(S(G, t)) = \chi(G).$$

*Proof.* Let  $w$  be a word of length  $t - 1$  on the alphabet  $V$ . By definition of  $S(G, t)$ , the subgraph  $\langle V_w \rangle$  of  $S(G, t)$  induced by the set  $V_w = \{wx : x \in V(G)\}$  is isomorphic to  $G$ . Hence,  $\chi(S(G, t)) \geq \chi(\langle V_w \rangle) = \chi(G)$ .

Now, let  $f : V \rightarrow \{1, 2, \dots, k\}$  be a proper vertex-colouring of  $G$  and let  $f_1 : V^t \rightarrow \{1, 2, \dots, k\}$  be a map defined by  $f_1(wx) = f(x)$ , for all  $wx \in V^t$ . If two vertices  $wx, w'y \in V^t$  are adjacent in  $S(G, t)$ , then  $x$  and  $y$  are adjacent in  $G$ . Hence, if  $wx, w'y \in V^t$  are adjacent in  $S(G, t)$ , then  $f_1(wx) = f(x) \neq f(y) = f_1(w'y)$  and, as a consequence,  $f_1$  is a proper vertex-colouring of  $S(G, t)$ . Therefore,  $\chi(S(G, t)) \leq \chi(G)$ .  $\square$

As a direct consequence of Theorem 5 we deduce the following result.

**Corollary 6.** *For any bipartite graph  $G$  and any positive integer  $t$ ,  $S(G, t)$  is bipartite.*

A *clique* of a graph  $G = (V, E)$  is a subset  $C \subseteq V$  such that for any pair of different vertices  $v, w \in C$ , there exists an edge  $\{v, w\} \in E$ , *i.e.*, the subgraph induced by  $C$  is complete. The *clique number* of a graph  $G$ , denoted by  $\omega(G)$ , is the number of vertices in a maximum clique of  $G$ . The chromatic number of a graph is equal to or greater than its clique number, *i.e.*,  $\chi(G) \geq \omega(G)$ .

It is well-known that the problem of finding a maximum clique is NP-complete, [2]. We shall show that the clique number of a generalized Sierpiński graph is equal to the clique number of its base graph.

**Theorem 7.** *For any graph  $G$  of order  $n$  and any positive integer  $t$ ,*

$$\omega(S(G, t)) = \omega(G).$$

*Proof.* We shall show that for any  $t \geq 2$ ,  $\omega(S(G, t)) = \omega(S(G, t-1))$ . Let  $x \in V$  and let  $\langle V_x \rangle$  be the subgraph of  $S(G, t)$  induced by the set  $V_x = \{xw : w \in V^{t-1}\}$ . Since  $\langle V_x \rangle \cong S(G, t-1)$ ,  $\omega(S(G, t)) \geq \omega(\langle V_x \rangle) = \omega(S(G, t-1))$ .

Now, let  $C$  be a maximum clique of  $S(G, t)$  and  $xw_1, yw_2 \in C$ . If  $x \neq y$ , then  $yw_2 = yxx \dots x$  is the only vertex not belonging to  $V_x$  which is adjacent to  $xw_1 = xyy \dots y$  and, analogously,  $xw_1 = xyy \dots y$  is the only vertex not belonging to  $V_y$  which is adjacent to  $yw_2 = yxx \dots x$ . Hence,  $x \neq y$  leads to  $|C| = 2 = \omega(S(G, t-1))$  and so  $|C| > 2$  leads to  $x = y$  which implies that  $C \subset V_x$ . Thus,  $\omega(S(G, t)) = |C| \leq \omega(\langle V_x \rangle) = \omega(S(G, t-1))$ .

Therefore,  $\omega(S(G, t)) = \omega(S(G, t-1)) = \dots = \omega(S(G, 1)) = \omega(G)$ . □

## 4 Vertex Cover Number and Independence Number

A *vertex cover* of a graph  $G$  is a set of vertices such that each edge of  $G$  is incident to at least one vertex of the set. The *vertex cover number* of  $G$ , denoted by  $\beta(G)$ , is the smallest cardinality of a vertex cover of  $G$ . For example, in the graph  $G$  of Figure 1,  $\{3, 4, 6\}$  is a vertex cover of minimum cardinality and so  $\beta(G) = 3$ .

It is well-known that the problem of finding a minimum vertex cover is a classical optimization problem in computer science and is a typical example of an NP-hard optimization problem, [2]. As the next result shows, the vertex cover number of a generalized Sierpiński graph can be computed from the vertex cover number and the order of the base graph.

**Theorem 8.** *For any graph  $G$  of order  $n$  and any positive integer  $t$ ,*

$$\beta(S(G, t)) = n^{t-1}\beta(G).$$

*Proof.* Let  $w \in V^{t-1}$  be a word of length  $t - 1$  on the alphabet  $V$ . By definition of  $S(G, t)$ , the subgraph  $\langle V_w \rangle$  of  $S(G, t)$  induced by the set  $V_w = \{wx : x \in V\}$  is isomorphic to  $G$ . Hence,

$$\beta(S(G, t)) \geq \sum_{w \in V^{t-1}(G)} \beta(\langle V_w \rangle) = n^{t-1} \beta(G).$$

Now, let  $C \subset V$  be a vertex cover of  $G$  of cardinality  $|C| = \beta(G)$  and let

$$C' = \{wv : v \in C \text{ and } w \in V^{t-1}\}.$$

Since  $\langle V_w \rangle \cong G$ , for any  $w \in V^{t-1}$  we have that  $C'_w = \{wv : v \in C\} \subset C'$  is a vertex cover of  $\langle V_w \rangle$ . In addition, if two vertices  $w_1y, w_2x \in V^t$ ,  $w_1 \neq w_2$ , are adjacent in  $S(G, t)$ , then  $x$  and  $y$  are adjacent in  $G$  and so  $x \in C$  or  $y \in C$ . Hence,  $w_1y \in C'$  or  $w_2x \in C'$ . Therefore,  $C'$  is a vertex cover of  $S(G, t)$  and, as a consequence,  $\beta(S(G, t)) \leq |C'| = n^{t-1}|C| = n^{t-1}\beta(G)$ .  $\square$

Recall that the largest cardinality of a set of vertices of  $G$ , no two of which are adjacent, is called the *independence number* of  $G$  and is denoted by  $\alpha(G)$ . For example, in the graph  $G$  of Figure 1,  $\{1, 2, 5, 7\}$  is an independent set of maximum cardinality and so  $\alpha(G) = 4$ .

The following well-known result, due to Gallai, states the relationship between the independence number and the vertex cover number of a graph. Such a result will provide us with another very useful result on generalized Sierpiński graphs.

**Theorem 9.** [1] *For any graph  $G$  of order  $n$ ,  $\beta(G) + \alpha(G) = n$ .*

By using this result and Theorem 8 we obtain a formula for the independence number of  $S(G, t)$ .

**Theorem 10.** *For any graph  $G$  of order  $n$  and any positive integer  $t$ ,*

$$\alpha(S(G, t)) = n^{t-1} \alpha(G).$$

## 5 Domination Number

For a vertex  $v$  of  $G = (V, E)$ ,  $N_G(v)$  denotes the set of neighbours that  $v$  has in  $G$ . A set  $D \subseteq V$  is *dominating* in  $G$  if every vertex of  $V - D$  has at least one neighbour in  $D$ , i.e.,  $D \cap N_G(u) \neq \emptyset$ , for all  $u \in V - D$ . The *domination number* of  $G$ , denoted by  $\gamma(G)$ , is the minimum cardinality among all dominating sets in  $G$ . A dominating set of cardinality  $\gamma(G)$  is called a  $\gamma(G)$ -set. Let  $\mathcal{D}(G)$  be the set of all  $\gamma(G)$ -sets. The dominating set problem, which concerns testing whether  $\gamma(G) \leq k$  for a given graph  $G$  and input  $k$ , is a classical NP-complete decision problem in computational

complexity theory, [2]. In this section we obtain an upper bound on the domination number of  $S(G, t)$  and we show that the bound is tight.

We define the parameter  $\xi(G)$  as follows:

$$\xi(G) = \max_{D \in \mathcal{D}(G)} \{|D'| : D' \subseteq D \text{ and } \langle D' \rangle \text{ has no isolated vertices}\}.$$

Notice that  $0 \leq \xi(G) \leq \gamma(G)$ . In particular,  $\xi(G) = 0$  if and only if any  $\gamma(G)$ -set is independent, while  $\xi(G) = \gamma(G)$  if and only if there exists a  $\gamma(G)$ -set whose induced subgraph has no isolated vertices.

**Theorem 11.** *For any graph  $G$  of order  $n$  and any integer  $t \geq 2$ ,*

$$\gamma(S(G, t)) \leq n^{t-2}(n\gamma(G) - \xi(G)).$$

*Proof.* Let  $D$  be a  $\gamma(G)$ -set and let

$$D_{t-1} = \{wx : w \in V^{t-1} \text{ and } x \in D\}.$$

If  $u \in V$  is adjacent to  $v \in D$ , then for any  $w \in V^{t-1}$ , we have that  $wu \in V^t$  is adjacent to  $wv \in D_{t-1}$  and, as a consequence,  $D_{t-1}$  is a dominating set in  $S(G, t)$ .

Now, assume that there exists  $D' \subseteq D$  such that the subgraph induced by  $D'$  has no isolated vertices and  $|D'| = \xi(G)$ . Let define the set

$$D_{t-2} = \{w'uu : w' \in V^{t-2} \text{ and } u \in D'\}.$$

We shall show that  $D^* = D_{t-1} - D_{t-2}$  is a dominating set in  $S(G, t)$ . To this end, taking  $w' \in V^{t-2}$  and  $u \in D'$ , we only need to show that each  $x \in N_{S(G, t)}(w'uu) \cup \{w'uu\}$  is dominated by some vertex belonging to  $D^*$ . Since  $w'uu$  is dominated by  $w'uv \in D^*$ , for some  $v \in D' \cap N_G(u)$ , from now on we assume that  $x \in N_{S(G, t)}(w'uu)$ . Now, if  $x = w'uz$ , for some  $z \in N_G(u)$ , then  $x$  is dominated by  $w'zu \in D^*$ , so we assume that  $x = w''zz$ , where  $w'' \in V^{t-2}$  and  $z \in N_G(u)$ . Thus, for  $z \notin D$  or  $z \in D \cap D'$  we have that  $x = w''zz$  is dominated by  $w''zu \in D^*$ . Finally, if  $z \in D - D'$ , then  $x = w''zz \in D^*$ .

Hence,  $D_{t-1} - D_{t-2}$  is a dominating set in  $S(G, t)$  and, as a consequence,

$$\gamma(S(G, t)) \leq |D_{t-1} - D_{t-2}| = n^{t-1}|D| - n^{t-2}|D'|.$$

Therefore, the result follows. □

From now on  $\Omega(G)$  denotes the set of vertices of degree one in  $G$ .

**Lemma 12.** *Let  $G$  be a graph such that  $\gamma(G) = \beta(G)$ . If there exists a unique  $\gamma(G)$ -set  $D$ , then  $|\Omega(G) \cap N_G(x)| \geq 2$ , for every  $x \in D$ .*



*Proof.* Let  $D$  be a  $\beta(G)$ -set. Since  $\gamma(G) = \beta(G)$ ,  $D$  is a  $\gamma(G)$ -set and  $V - D$  is an  $\alpha(G)$ -set. Assume that  $D$  is the only  $\gamma(G)$ -set and suppose that  $|\Omega(G) \cap N_G(v)| \leq 1$ , for some  $v \in D$ . Let  $x \in N_G(v) - D$ . If  $x \notin \Omega(G)$ , then there exists  $v' \in D$  such that  $x \in N_G(v')$ . Hence, if  $\Omega(G) \cap N_G(v) = \emptyset$ , then  $(D - \{v\}) \cup \{x\}$  is a dominating set, which is a contradiction. Also, if  $\Omega(G) \cap N_G(v) = \{y\}$ , then  $(D - \{v\}) \cup \{y\}$  is a dominating set, which is a contradiction again. Therefore, the result follows.  $\square$

**Theorem 13.** *Let  $G$  be a graph of order  $n$  such that there exists a unique  $\gamma(G)$ -set and  $\gamma(G) = \beta(G)$ . Then for any integer  $t \geq 2$*

$$\gamma(S(G, t)) = n^{t-2}(n\gamma(G) - \xi(G)).$$

*Proof.* Let  $D \subset V$  be the only  $\gamma(G)$ -set. As we have shown in the proof of Theorem 11,  $D^* = D_{t-1} - D_{t-2}$  is a dominating set of  $S(G, t)$  and  $|D^*| = n^{t-2}(n\gamma(G) - \xi(G))$ . Let  $D^\pi$  be a dominating set of  $S(G, t)$  of minimum cardinality. If  $D^* - D^\pi = \emptyset$ , then  $|D^*| \leq |D^\pi|$  and, as a consequence,  $\gamma(S(G, t)) = |D^*|$ . Let  $w \in V^{t-1}$  and assume that  $wx \in D^* - D^\pi$ . Since  $x \in D$ , by Lemma 12 we have  $|\Omega(G) \cap N_G(x)| \geq 2$ . Thus, if  $wx$  is not the extreme of  $\langle V_w \rangle$ , then there are at least two vertices of degree one adjacent to  $wx$  in  $S(G, t)$ , which implies that  $wx \in D^\pi$  and it is a contradiction. Thus,  $wx$  is the extreme vertex of  $\langle V_w \rangle$  i.e.,  $wx = w'xx$ , for some  $w' \in V^{t-2}$ . Notice that:

- $w'xx$  is dominated by a vertex  $w''y \in D^\pi$ . Now, if  $w''y \in D^*$ , then  $x$  and  $y$  are adjacent in  $G$ ,  $x \in D'$  and  $w'xx \notin D^*$ , which is a contradiction. Hence,  $w''y \in D^\pi - D^*$ .
- The only vertex in  $D^* - D^\pi$  dominated by  $w''y$  is  $w'xx$ , as a vertex in  $S(G, t)$  can only be adjacent to one extreme vertex.

Let  $S \subseteq D^\pi - D^*$  such that each vertex of  $D^* - D^\pi$  is dominated by a vertex in  $S$ . Then we can define a mapping  $f : S \rightarrow D^* - D^\pi$ , where  $f(u) = v$  means that  $v$  is dominated by  $u$ . As  $f$  is an onto mapping,  $|D^* - D^\pi| \leq |S| \leq |D^\pi - D^*|$ . Therefore,  $|D^*| \leq |D^\pi|$ , and we can conclude that  $\gamma(S(G, t)) = |D^\pi| = |D^*| = n^{t-2}(n\gamma(G) - \xi(G))$ .  $\square$

**Lemma 14.** *Let  $G$  be a graph of order  $n$  and let  $t \geq 3$  be an integer. If  $\gamma(S(G, t)) = n^{t-1}\gamma(G)$ , then there exists a unique  $\gamma(G)$ -set.*

*Proof.* Assume that  $\gamma(S(G, t)) = n^{t-1}\gamma(G)$ . Notice that, by Theorem 11, we have that  $\xi(G) = 0$ . Suppose, for contradiction proposes, that  $A$  and  $B$  are two different  $\gamma(G)$ -sets. In such a case, there exist  $a \in A - B$  and  $b \in B$  such that  $b$  dominates  $a$ . Now, if  $b \in A$ , then  $\xi(G) > 0$ , which is a contradiction. So,  $b \notin A$ . Following a procedure analogous to that used in the proof of Theorem 11 we see that  $A_{t-1} = \{wv : w \in V^{t-1} \text{ and } v \in A\}$  is a dominating set of  $S(G, t)$ . Hence,

$$A' = (A_{t-1} - (\{abb \dots bx : x \in A\} \cup \{baa \dots a\})) \cup \{abb \dots bx : x \in B\}$$

is a dominating set of  $S(G, t)$  as any vertex in  $\{abb \dots bx : x \in V\}$  is dominated by some vertex in  $\{abb \dots bx : x \in B\} \subset A'$ ,  $baa \dots a$  is dominated by  $abb \dots b \in A'$  and, for any  $z \in N_G(a)$ ,  $baa \dots az$  is dominated by  $baa \dots aza \in A'$ . Thus,  $\gamma(S(G, t)) \leq |A'| = n^{t-1}\gamma(G) - 1$ , which is a contradiction. Therefore, the result follows.  $\square$

**Theorem 15.** *Let  $G$  be a graph of order  $n$  such that  $\gamma(G) = \beta(G)$  and let  $t \geq 3$  be an integer. The following assertions are equivalent.*

- (a)  $\gamma(S(G, t)) = n^{t-1}\gamma(G)$ .
- (b)  $\xi(G) = 0$  and there exists a unique  $\gamma(G)$ -set.

*Proof.* Assume that  $\gamma(S(G, t)) = n^{t-1}\gamma(G)$ . By Theorem 11, we have that  $\xi(G) = 0$  and by Lemma 14 we have that there exists a unique  $\gamma(G)$ -set.

Now, if  $\xi(G) = 0$  and there exists a unique  $\gamma(G)$ -set, then by Theorem 13 we conclude that  $\gamma(S(G, t)) = n^{t-1}\gamma(G)$ .  $\square$

It is ready to see that  $\gamma(S(K_{1,r}, 2)) = r + 1$ . Hence, from Theorem 15 we deduce the following result.

**Corollary 16.** *For any positive integers  $r$  and  $t$ ,*

$$\gamma(S(K_{1,r}, t)) = (r + 1)^{t-1}.$$

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